also some approximate approaches to the construction of the models of regular structures.

Thus, we have proved the –

Theorem. The deformation of an arbitrary regular elastic structure, possessing the property of quasi-periodicity of the displacements, is identical "in the large" with the deformation of the homogeneous anisotropic medium, characterized by the relations (5.7).

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ECHO SIGNAL OF A FINITE SPHERICAL PULSE FROM AN ELASTIC CYLINDRICAL SHELL

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An approximate method of calculating the echo signal of a finite, centrally-symmetric pressure pulse from an infinite elastic cylindrical shell in an infinite ideal compressible fluid is proposed. The shell motion is described by using a linear shell theory of Timoshenko type. The problem is solved by a triple application of integral transformations (in time and the longitudinal coordinate, a Fourier transform, and in the polar angle, a Sommerfeld-Watson transform).

The nonstationary interaction of spherical pressure pulses in a fluid with an elastic cylindrical shell has been studied in [1 – 3], where a Laplace time transformation, a Fourier transformation in the longitudinal coordinate, and either a Fourier series expansion [1, 3] or a Fourier transformation [2] in the polar angle have been used to solve the problem. However, calculation of the rapidly varying components of the Fourier-series solution id difficult because of the slow convergence. Difficulties in inverting the transform appear in the application of the Fourier transform in the polar angle; the saddle point method used in [2] does not permit taking correct account of the influence of the elastic waves being propagated in the shell.

A Sommerfeld-Watson transformation in the polar angle, which admits of a more exact inversion, is applied herein. A Fourier transform in time has been chosen instead of the Laplace tranform used in [1−3]. The solution in the space of the Fourier time transform can be treated as the solution of the corresponding stationary problem, and this permits using the experience acquired in solving stationary problems, and analyzing the solution in transform space.

This paper is an extension of the method of compiling the algorithm developed on the basis of [4−6] to a cylindrical shell (*).

1. Formulation of the problem. Let R, ϑ, Ξ be cylindrical coordinates, t the time ($t = 0$ when the beginning of the pulse emerges from the source), P_i (R, ϑ, Ξ, t) the incident pressure pulse, P_e (R, ϑ, Ξ, t) the scattered pressure field (echo signal), c the speed of sound in the fluid, ρ the fluid density, L the distance from the center of the source, R_0 the distance between the center of the source and the shell axis, R_k, h the middle surface radius and the shell thickness, respectively, E, ν, ρ_1 the elastic modulus, the Poisson's ratio, and the density of the shell material, respectively, u, v, w the displacements, ψ_ξ, ψ_ϑ the angles of rotation in a Timoshenko type shell theory, Q the normal pressure acting on the shell, and t_p the incident pulse duration.

Let us use the dimensionless cylindrical coordinates

$$r = R / R_k, \quad \vartheta, \quad \xi = \Xi / R_k, \quad \tau = ct / R_k \tag{1.1}$$

and the dimensionless quantities

$$p_i(r, \vartheta, \xi, \tau) = \frac{P_i(R, \vartheta, \Xi, t)}{\rho c^2}, \quad p_e(r, \vartheta, \xi, \tau) = \frac{P_e(R, \vartheta, \Xi, t)}{\rho c^2}$$

$$q = \frac{Q}{\rho c^2}, \quad a^2 = \frac{h^2}{12 R_k^2}$$

$$\Phi_1 = \frac{u}{R_k}, \quad \Phi_2 = \psi_\xi, \quad \Phi_3 = \frac{v}{R_k}, \quad \Phi_4 = \psi_\vartheta, \quad \Phi_5 = \frac{w}{R_k}, \quad l = \frac{L}{R_k}, \quad r_0 = \frac{R_0}{R_k}$$

$$\alpha = \frac{1-\nu}{2}, \quad \beta = \frac{(1-\nu^2)\rho_1 c^2}{E}, \quad \varkappa = \frac{h\rho_1}{R_k \rho \beta}, \quad \tau_p = \frac{ct_p}{R_k} \tag{1.2}$$

On the basis of geometry

$$l = (\xi^2 + y^2)^{1/2}, \qquad y = (r^2 + r_0^2 - 2rr_0 \cos\vartheta)^{1/2} \tag{1.3}$$

Let a source with center O (r_0, 0, 0) radiate a finite spherical pressure pulse

$$p_i = A_0 l^{-1} f(\tau - l)\,[H(\tau - l) - H(\tau - l - \tau_p)] \tag{1.4}$$

in an ideal compressible fluid. Here A_0 is a constant governing the pulse amplitude, f is the law of pressure variation in the pulse, H is the Heaviside function. The pulse

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*) Metsaveer, Ia. A., Algorithm to calculate echo signals from an elastic spherical shell in a fluid by summing separate groups of travelling waves. Preprint №3, Inst. of Cybernetics ESSR Acad. of Science, Tallin, 1971.

(1.4) is incident on a shell with middle surface $r = 1$, and being dissipated theoron, generating elastic waves in the shell which, in turn, excite a radiated pressure field in the surrounding medium.

To describe the elastic waves in the shell, let us use equations of a Timoshenko-type shell theory [7]

$$L_{ij}\Phi_j = -\delta_{i5}\varkappa^{-1}q \qquad i=1,2,\ldots,5; \quad j=1,2,\ldots,5 \tag{1.5}$$

$$L_{11} = \frac{\partial^2}{\partial\xi^2} + (1+a^2)\alpha\frac{\partial^2}{\partial\vartheta^2} - \beta\frac{\partial^2}{\partial\tau^2}, \quad L_{12} = a^2\left(\frac{\partial^2}{\partial\xi^2} - \alpha\frac{\partial^2}{\partial\vartheta^2} - \beta\frac{\partial^2}{\partial\tau^2}\right)$$

$$L_{13} = \frac{1+\nu}{2}\frac{\partial^2}{\partial\xi\partial\vartheta}, \quad L_{14} = 0, \quad L_{15} = \nu\frac{\partial}{\partial\xi}, \quad L_{21} = a^2\left(\frac{\partial^2}{\partial\xi^2} - \alpha\frac{\partial^2}{\partial\vartheta^2} - \beta\frac{\partial^2}{\partial\tau^2}\right)$$

$$L_{22} = a^2\left(\frac{\partial^2}{\partial\xi^2} + \alpha\frac{\partial^2}{\partial\vartheta^2} - \beta\frac{\partial^2}{\partial\tau^2}\right) - k_\xi^2\alpha, \quad L_{23} = 0, \quad L_{24} = a^2\frac{1+\nu}{2}\frac{\partial^2}{\partial\xi\partial\vartheta}$$

$$L_{25} = -k_\xi^2\alpha\frac{\partial}{\partial\xi}, \quad L_{31} = \frac{1+\nu}{2}\frac{\partial^2}{\partial\xi\partial\vartheta}, \quad L_{32} = 0, \quad L_{33} = \alpha\frac{\partial^2}{\partial\xi^2} + (1+a^2)\left(\frac{\partial^2}{\partial\vartheta^2} - \right.$$

$$\left. k_\vartheta^2\alpha\right) - \beta\frac{\partial^2}{\partial\tau^2}, \quad L_{34} = a^2\left(\alpha\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\vartheta^2} + k_\vartheta^2\alpha - \beta\frac{\partial^2}{\partial\tau^2}\right) + k_\vartheta^2\alpha$$

$$L_{35} = (1+a^2)(1+k_\vartheta^2\alpha)\frac{\partial}{\partial\vartheta}, \quad L_{41} = 0, \quad L_{42} = \frac{1+\nu}{2}a^2\frac{\partial^2}{\partial\xi\partial\vartheta}$$

$$L_{43} = a^2\left(\alpha\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\vartheta^2} + k_\vartheta^2\alpha - \beta\frac{\partial^2}{\partial\vartheta^2}\right) + k_\vartheta^2\alpha, \quad L_{44} = a^2\left(\alpha\frac{\partial^2}{\partial\xi^2} + \right.$$

$$\left.\frac{\partial^2}{\partial\vartheta^2} - k_\vartheta^2\alpha - \beta\frac{\partial^2}{\partial\tau^2}\right) - k_\vartheta^2\alpha, \quad L_{45} = -[(1+a^2)k_\vartheta^2\alpha + a^2]\frac{\partial}{\partial\vartheta}$$

$$L_{51} = -\nu\frac{\partial}{\partial\xi}, \quad L_{52} = k_\xi^2\alpha\frac{\partial}{\partial\xi}, \quad L_{53} = -(1+a^2)(1+k_\vartheta^2\alpha)\frac{\partial}{\partial\vartheta}$$

$$L_{54} = [(1+a^2)k_\vartheta^2\alpha + a^2]\frac{\partial}{\partial\vartheta}, \quad L_{55} = k_\xi^2\alpha\frac{\partial^2}{\partial\xi^2} +$$

$$(1+a^2)\left(k_\vartheta^2\alpha\frac{\partial^2}{\partial\vartheta^2} - 1\right) - \beta\frac{\partial^2}{\partial\tau^2}, \quad k_\vartheta^2 = k_\xi^2 = \frac{\pi^2}{12}$$

The scattered pressure field p_e, consisting of the field reflected from the shell surface and the radiated field, should satisfy a wave equation as does the incident pulse (1.4)

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\tau^2}\right]p_e = 0 \tag{1.6}$$

Let us take the following contact conditions on the middle surface:

$$[p_i + p_e]_{r=1} = -q, \qquad [\partial p_i/\partial r + \partial p_e/\partial r]_{r=1} = -\partial^2\Phi_5/\partial\tau^2 \tag{1.7}$$

Since the Fourier transform in time is used, the initial conditions are not formulated in the customary sense, but all the components of the solution are assumed to be zero as

$$\tau \to \pm\infty$$

2. Solution in the transform space. Let us define the Fourier transformations in the time τ and the longitudinal coordinate ξ by means of the formulas

$$p(r,\vartheta,\xi,\tau) = \int_{-\infty}^{\infty} p^F(r,\vartheta,\xi;x)e^{-ix\tau}dx$$

$$p^F(r,\vartheta,\xi;x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(r,\vartheta,\xi,\tau)e^{ix\tau}d\tau \tag{2.1}$$

$$p^F(r, \vartheta, \xi; x) = \int_{-\infty}^{\infty} p^{FF}(r, \vartheta; \lambda, x) e^{-i\lambda\xi} d\lambda$$

$$p^{FF}(r, \vartheta; \lambda, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p^F(r, \vartheta, \xi; x) e^{i\lambda\xi} d\xi \qquad (2.2)$$

On the basis of (2.1) we have

$$p_i^F = A_0 l^{-1} g(x) e^{ilx} \qquad (2.3$$

$$g(x) = \frac{1}{2\pi} \int_0^{\tau_p} f(\tau) e^{ix\tau} d\tau \qquad (2.4)$$

Furthermore, transforming (2.3) by means of (2.2), taking account of (1.3), we obtain

$$p_i^{FF} = {}^1/_2 i A_0 g(x) H_0^{(1)}(y\eta), \qquad \eta = (x^2 - \lambda^2)^{1/2} \qquad (2.5)$$

Here $H_0^{(1)}$ is the zero order Hankel function of the first kind. Using the "addition theorem" for $H_0^{(1)}$, let us represent (2.5) as (J_m is the Bessel function)

$$p_i^{FF} = \frac{1}{2} i A_0 g(x) \sum_{m=0}^{\infty} \varepsilon_m H_m^{(1)}(\eta r_0) J_m(\eta r) \cos m\vartheta, \qquad \varepsilon_m = 2 - \delta_{m0} \quad (2.6)$$

The echo signal in the space of the double Fourier transform can also be represented in the form of series. Taking account of (2.6) and the radiation condition at infinity [8]

$$\lim_{r\to\infty} r^{1/2}(\partial p_e^{FF} / \partial r - i\eta p_e^{FF}) = 0 \qquad (2.7)$$

let us seek the transformed echo signal in the form

$$p_e^{FF} = \frac{1}{2} i A_0 g(x) \sum_{m=0}^{\infty} \varepsilon_m H_m^{(1)}(\eta r_0) b_m H_m^{(1)}(\eta r) \cos m\vartheta \qquad (2.8)$$

Here b_m is to be calculated on the basis of (1.5) and the contact conditions (1.7). The solution of the system (1.5) in the space of the double Fourier transform can be represented as

$$\Phi_1^{FF} = \sum_{m=0}^{\infty} \Psi_{1m} \cos m\vartheta, \qquad \Phi_2^{FF} = \sum_{m=0}^{\infty} \Psi_{2m} \cos m\vartheta$$

$$\Phi_3^{FF} = \sum_{m=0}^{\infty} \Psi_{3m} \sin m\vartheta, \qquad \Phi_4^{FF} = \sum_{m=0}^{\infty} \Psi_{4m} \sin m\vartheta$$

$$\Phi_5^{FF} = \sum_{m=0}^{\infty} \Psi_{5m} \cos m\vartheta \qquad (2.9)$$

Substituting (2.6), (2.8), (2.9) into (1.5) and the contact condition (1.7) after transformation by means of (2.1), (2.2) in the double Fourier transform space, we obtain

$$b_m = -\frac{x^2 A_{55} J_m(\eta) - \eta\varkappa D \partial J_m(\eta) / \partial\eta}{x^2 A_{55} H_m^{(1)}(\eta) - \eta\varkappa D\, \partial H_m^{(1)}(\eta) / \partial\eta}, \qquad m = 1, 2, \ldots \qquad (2.10)$$

$$D = \det | a_{ij} |, \qquad i = 1, 2, \ldots, 5; \; j = 1, 2, \ldots, 5$$

$$a_{11} = \lambda^2 + (1 + a^2)\alpha m^2 - \beta x^2, \quad a_{12} = a^2(\lambda^2 - \alpha m^2 - \beta x^2)$$

$$a_{13} = -i\lambda m\,(1+\nu)\,/\,2,\quad a_{14} = 0,\quad a_{15} = -i\nu\lambda,\quad a_{21} = a^2\,(\lambda^2 - \alpha m^2 - \beta x^2),\quad a_{22} = a^2\,(\lambda^2 + \alpha m^2 - \beta x^2) + k_\xi{}^2\alpha,\quad a_{23} = 0$$

$$a_{24} = -i\lambda m a^2\,(1+\nu)\,/\,2,\quad a_{25} = k_\xi{}^2\alpha i\lambda,\quad a_{31} = -i\lambda m\,(1+\nu)\,/\,2$$

$$a_{32} = 0,\quad a_{33} = -\alpha\lambda^2 - (1+a^2)\,(m^2 + k_\vartheta{}^2\alpha) + \beta x^2,\quad a_{34} = a^2\,(m^2 - \alpha\lambda^2 + k_\vartheta{}^2\alpha + \beta x^2) + k_\vartheta{}^2\alpha,\quad a_{35} = -(1+a^2)\,(1+k_\vartheta{}^2\alpha)\,m$$

$$a_{41} = 0,\quad a_{42} = -i\lambda m a^2\,(1+\nu)\,/\,2,\quad a_{43} = a^2\,(m^2 - \alpha\lambda^2 + k_\vartheta{}^2\alpha + \beta x^2) + k_\vartheta{}^2\alpha,\quad a_{44} = -a^2\,(\alpha\lambda^2 + m^2 + k_\vartheta{}^2\alpha - \beta x^2) - k_\vartheta{}^2\alpha$$

$$a_{45} = [(1+a^2)\,k_\vartheta{}^2\alpha + a^2]m,\quad a_{51} = -i\nu\lambda,\quad a_{52} = k_\xi{}^2\alpha i\lambda$$

$$a_{53} = -(1+a^2)\,(1+k_\vartheta{}^2\alpha)\,m,\quad a_{54} = [(1+a^2)\,k_\vartheta{}^2\alpha + a^2]\,m$$

$$a_{55} = -k_\xi{}^2\alpha\lambda^2 - (1+a^2)\,(k_\vartheta{}^2\alpha m^2 + 1) + \beta x^2$$

Here A_{55} is the corresponding cofactor of the determinant D.

The series (2.8) is summed by using the Sommerfeld-Watson transform [9]

$$\sum_{m=0}^{\infty} \varepsilon_m F_m \cos m\vartheta = i \int_\Gamma F_\mu \sin^{-1}\mu\pi \cos\mu\,(\pi - \vartheta)\,d\mu \tag{2.11}$$

The contour of integration Γ which passes through the origin includes the positive part of the real axis in a clockwise direction on the complex μ plane. Taking account of (2.11), the transformed echo signal (2.8) can be rewritten as the integral

$$p_e^{FF} = -\frac{1}{2} A_0 g\,(x) \int_\Gamma b_\mu H_\mu^{(1)}(\eta r_0)\, H_\mu^{(1)}(\eta r) \sin^{-1}\mu\pi \cos\mu\,(\pi - \vartheta)\,d\mu \tag{2.12}$$

Using the relationship
$$J_\mu(\eta) = {}^1/_2\,[H_\mu^{(1)}(\eta) + H_\mu^{(2)}(\eta)]$$

the integral (2.12) can be rewritten as

$$p_e^{FF} = \frac{1}{4} A_0 g\,(x) \int_\Gamma F_1^{-1} F_2 R_\mu \sin^{-1}\mu\pi \cos\mu\,(\pi - \vartheta)\,d\mu \tag{2.13}$$

Here

$$F_{1,2} = x^2 A_{55} - \eta\varkappa D\,[\partial H_\mu^{(1,2)}(\eta)\,/\,\partial\eta]\,[H_\mu^{(1,2)}(\eta)]^{-1} \tag{2.14}$$

$$R_\mu = H_\mu^{(1)}(\eta r_0)\, H_\mu^{(1)}(\eta r)\, H_\mu^{(2)}(\eta)\,[H_\mu^{(1)}(\eta)]^{-1} \tag{2.15}$$

It is necessary to replace m everywhere by μ in the elements of the determinant D when evaluating $F_{1,2}$, where μ is a complex number.

Let us turn attention to the fact that the Hankel functions in (2.13) are partially collected in R_μ and partially in $F_{1,2}$. Consequently, the function R_μ is related just to the shell geometry, and the function $F_{1,2}$ to its elasticity. In the limit case when $\varkappa = 0$, the ratio $F_1{}^{-1}F_2$ equals unity which corresponds to the case of an "acoustically soft" cylinder.

Now using the relationships [10]

$$\cos\mu\,(\pi - \vartheta) = {}^1/_2\,[e^{i\mu(\pi-\theta)} + e^{-i\mu(\pi-\theta)}]$$

$$\sin^{-1}\mu\pi = -2ie^{i\mu\pi} \sum_{n=0}^{\infty} e^{2i\mu\pi n}$$

and the definitions (2.1), (2.2), taking account of (2.13), the echo signal can be represented as

$$p_e = -\frac{1}{4} iA_0 \sum_{k=1}^{2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g(x) \int_{-\infty}^{\infty} \int_{\Gamma} F_1^{-1} F_2 R_\mu e^{i(\mu\vartheta_{nk} - \lambda\xi - x\tau)} d\mu d\lambda dx$$

$$\vartheta_{n1} = \vartheta + 2n\pi, \qquad \vartheta_{n2} = 2\pi - \vartheta + 2n\pi \tag{2.16}$$

3. Inversion of the Fourier transformation in the longitudinal coordinate. Let us represent the echo signal (2.16) as

$$p_e = -\frac{1}{4} iA_0 \sum_{k=1}^{2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g(x) \int_{\Gamma} S_{nk} e^{-ix\tau} d\mu dx \tag{3.1}$$

$$S_{nk} = \int_{-\infty}^{\infty} F_1^{-1} F_2 R_\mu \exp[i(\mu\vartheta_{nk} - \lambda\xi)] d\lambda \tag{3.2}$$

To evaluate the integral (3.2), let us replace the Hankel functions in R_μ by their asymptotic Debye representations [9]

$$H_\mu^{(1,2)}(\eta) \approx (2/\pi\eta)^{1/2} (1 - z^2)^{-1/4} \exp[\pm i(\sigma - \pi/4)] \tag{3.3}$$
$$\sigma = \eta[(1 - z^2)^{1/2} - z \arccos z], \qquad z = \mu\eta^{-1}$$

Substituting (3.3) into (2.15) we have

$$R_\mu \approx (2/\pi\eta)(r_0 r)^{-1/2} D(z) \exp\{i\eta[d(z) + z h(z)]\} \tag{3.4}$$

Here

$$D(z) = [(1 - z^2 r_0^{-2})(1 - z^2 r^{-2})]^{-1/4}, \; d(z) = (r_0^2 - z^2)^{1/2} + (r^2 - z^2)^{1/2} - 2(1 - z^2)^{1/2} \tag{3.5}$$
$$h(z) = 2\arccos z - \arccos(z r_0^{-1}) - \arccos(z r^{-1})$$

For $r_0 \gg 1$, $r \gg 1$ formula (3.5) simplifies to the form

$$D(z) = 1, \quad d(z) = r_0 + r - 2(1 - z^2)^{1/2}, \quad h(z) = 2\arccos z - \pi \tag{3.6}$$

It is proposed to evaluate the Hankel functions in (2.14) by more exact methods. This recommendation is based on the following considerations. For very low and very high frequencies, the influence of the elastic properties of the scatterer on the scattered pressure field turn out to be immaterial [10]. The application of asymptotic represenrations which are valid for either low or high frequencies, will only give approximate results in the computation of the function R_μ and in the computation of F_1, F_2, incorrect results are possible.

Substituting (3.4) into the integral (3.2), taking account of the representation for η in (2.5), we have

$$S_{nk} = \frac{2}{\pi} \frac{D(z)}{\sqrt{r_0 r}} \int_{-\infty}^{\infty} \frac{F_2}{\eta F_1} \exp\{i[(x^2 - \lambda^2)^{1/2} \varphi_{nk}(z) - \lambda\xi]\} d\lambda$$

$$\varphi_{nk}(z) = d(z) + z[h(z) + \vartheta_{nk}] \tag{3.7}$$

Let us invert the integral (3.7) by the saddle point method. The saddle point has the coordinate

$$\lambda_{nk} = -x\xi[\varphi_{nk}^2(z) + \xi^2]^{-1/2}$$

and by using the standard technique we obtain

$$S_{nk} = \frac{2D(z)}{\sqrt{r_0 r}} \left(\frac{2x}{i\pi}\right)^{1/2} \frac{\varphi_{nk}(z)}{\psi_{nk}^{3/2}(z)} \left[\frac{F_2}{\eta F_1}\right]_{\lambda=\lambda_{nk}} e^{ix\psi_{nk}(z)} \tag{3.8}$$

$$\psi_{nk}(z) = [\varphi_{nk}^2(z) + \xi^2]^{1/2}$$

4. Inversion of the Sommerfeld-Watson transform. Taking account of (3.8) and the relationship for z from (3.3), let us give the echo-signal (3.1) the form

$$p_e = A_0 (2r_0 r)^{-1/2} \sum_{k=1}^{2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} g(x) M_{nk} e^{-ix\tau} dx \tag{4.1}$$

$$M_{nk} = -\left(\frac{ix}{\pi}\right)^{1/2} \int_{\Gamma_z} \frac{D(z)\varphi_{nk}(z)}{\psi_{nk}^{3/2}(z)} \left[\frac{F_2}{F_1}\right]_{\lambda=\lambda_{nk}} e^{ix\psi_{nk}(z)} dz \tag{4.2}$$

Here the contour Γ_z on the z plane corresponds to the contour Γ on the μ plane. In order to evaluate the contour integral (4.2) it is expedient to deform the contour of integration Γ_z so that the main contribution of the integral is determined by the portion of the contour enclosing the pole in the first quadrant of the z plane [11]. The initial contour of integration Γ_z and the deformed contour consisting of the sections Γ_0, Γ_∞, Γ_p, are represented in Fig. 1. It is known from [11] that the contributions of integration over the sections Γ_0, Γ_∞ of the deformed contour are small and can be neglected. Consequently, the main contribution is determined by integration over the section Γ_p of the deformed contour.

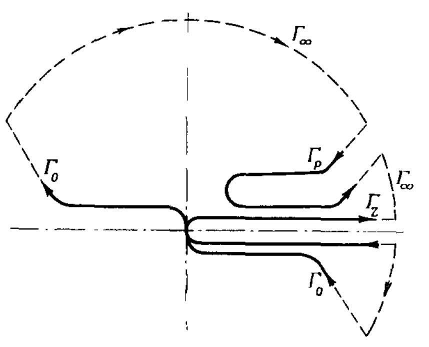

Fig. 1

To invert the integral (4.2), let us study the presence of saddle points. The coordinates z_0 of the saddle points are determined as the solution of the equation

$$h(z_0) + \vartheta_{nk} = 0 \tag{4.3}$$

This equation has a solution only for

$$\vartheta_{nk} \leqslant \vartheta_0, \qquad \vartheta_0 = \arccos(r_0^{-1}) + \arccos(r^{-1})$$

The integral governing the contribution from the saddle point can be represented as

$$M_0 = -\left(\frac{ix}{\pi}\right)^{1/2} \int_{\Gamma_p} \frac{D(z)\varphi_0(z)}{\psi_0^{3/2}(z)} \left[\frac{F_2}{F_1}\right]_{\lambda=\lambda_0} e^{ix\psi_0(z)} dz \tag{4.4}$$

$$\varphi_0(z) = d(z) + z[h(z) + \vartheta], \quad \psi_0(z) = [\varphi_0^2(z) + \xi^2]^{1/2}, \quad \lambda_0 = -x\xi\psi_0^{-1}(z)$$

The saddle point with coordinate z_0 is on the real axis of the z plane. For $\vartheta = 0$ it is at the point $z = 0$ and as ϑ grows from zero to ϑ_0 it moves to the right approaching the point $z = 1$. Since there is no $\sin^{-1}\mu\pi$ member in the integral (4.4), then the value of the integral does not change because the contour Γ_p does not cross the real axis of the z plane upon passing through the saddle point.

In evaluating the integral (4.4) by the saddle point method it must be taken into account that poles whose coordinates z_s are solutions of the equation $F_1 = 0$ are located in the first quadrant of the z plane, where F_1 is defined by (2.14). We select the method of evaluating the integral (4.4) depending on whether the saddle point is near one of the poles or not. If the saddle point is not near any of the poles, then evaluating the integral (4.4) by the ordinary saddle point method, we have

$$M_0 = -D(z_0)C(z_0)d^{1/2}(z_0)\psi_0^{-1}(z_0)(1-z_0^2)^{1/4}[F_1^{-1}F_2]_{\substack{\lambda=\lambda_0\\ z=z_0}}e^{ix\psi_0(z_0)}$$

$$C(z_0) = \{1 - {}^1/_2(1-z_0^2)^{1/2}[(r^2-z_0^2)^{-1/2} + (r_0^2-z_0^2)^{-1/2}]\},^{-1/2}$$

$$\psi_0(z_0) = [d^2(z_0)+\xi^2]^{1/2} \tag{4.5}$$

When the saddle point is near a pole with coordinate z_s, then we evaluate (4.4) by the method proposed in the foreword to [12]. To do this, let us replace the slowly varying integrand

$$E(z) = D(z)\varphi_0(z)\psi_0^{-3/2}(z)[F_2F_1^{-1}]_{\lambda=\lambda_0}$$

by its approximate expression

$$E(z) \approx E_0(z)(z-z_s)^{-1}, \quad E_0(z) = D(z)\varphi_0(z)\psi_0^{-3/2}(z)[(\partial F_1/\partial z)^{-1}F_2]_{\substack{\lambda=\lambda_0\\ z=z_s}} \tag{4.6}$$

After such a substitution, the integral (4.4) becomes

$$M_0 = -\left(\frac{ix}{\pi}\right)^{1/2}\int_{\Gamma_p} E_0(z)(z-z_s)^{-1}e^{ix\psi_0(z)}\,dz \tag{4.7}$$

When the pole with coordinate z_s is to the right of the contour of steepest descent passing through the saddle point parallel to the imaginary axis in the z plane, (4.7) is determined by the formula

$$M_0 = -i(ix\pi)^{1/2}E_0(z_0)W_T[k_0(z_s-z_0)]e^{ix\psi_0(z_0)}$$

$$W_T(z) = e^{-z^2}\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{\zeta^2}d\zeta\right] \tag{4.8}$$

$$k_0 = (ix)^{1/2}(1-z_0^2)^{-1/4}C^{-1}(z_0)d^{1/2}(z_0)\psi_0^{-1/2}(z_0)$$

(The function W_T is tabulated in [12]). When the saddle point is sufficiently far from the pole, then

$$W_T[k_0(z_s-z_0)] \approx i\pi^{-1/2}k_0^{-1}(z_s-z_0)^{-1} \tag{4.9}$$

Substituting (4.9) with (4.6) taken into account into (4.8), we obtain a result corresponding to (4.5) which has been obtained by the customary saddle point method. When the pole with coordinate z_s is to the left of the steepest descent contour of the saddle point method, evaluation of the integral (4.7) yields

$$M_0 = M_s + i(ix\pi)^{1/2}E_0(z_0)W_T[k_0(z_0-z_s)]e^{ix\psi_0(z_0)}$$

$$M_s = -2i(ix\pi)^{1/2}E_0(z_s)e^{ix\psi_0(z_s)} \tag{4.10}$$

Here M_s is the contribution to the integral (4.7) at the pole with coordinate z_s. For $\vartheta_{nk} > \vartheta_0$ the value of the integral (4.2) is determined as the sum of contributions at the poles with coordinate z_s in the first quadrant of the z plane

$$M_{nk} = -2i(ix\pi)^{1/2}D(z_s)\varphi_{nk}(z_s)\psi_{nk}^{-3/2}(z_s)[(\partial F_1/\partial z)^{-1}F_2]_{\substack{\lambda=\lambda_{nk}\\ z=z_s}}e^{ix\psi_{nk}(z_s)} \tag{4.11}$$

On the basis of the preceding, the echo signal (4.1) can be represented as

$$p_e = B_0 \int\limits_{-\infty}^{\infty} g(x) \left[p_0{}^F + \sum_{k=1}^{2} \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} p^F_{snk} \right] dx \tag{4.12}$$

$$B_0 = A_0 (2r_0 r)^{-1/2} \tag{4.13}$$

$$p_0{}^F = G_0(x) e^{-ix[\tau - \psi_0(z_0)]}, \quad p_{snk} = G_{snk}(x) e^{-ix[\tau - \psi_{nk}(z_s)]} \tag{4.14}$$

Depending on whether the saddle point is far from or near to the pole with coordinate z_s, the function $G_0(x)$ in (4.14) is evaluated by means of one of the following formulas:

$$G_0(x) = -D(z_0) C(z_0) d^{1/2}(z_0) \psi_0^{-1}(z_0) (1 - z_0{}^2)^{1/4} [F_1^{-1} F_2]_{\substack{\lambda=\lambda_0 \\ z=z_0}} \tag{4.15}$$

$$G_0(x) = -i(ix\pi)^{1/2} E_0(z_0) W_T(k_0 | z_s - z_0 |) \operatorname{sign}(\operatorname{Re} z_s - z_0) \tag{4.16}$$

The function $G_{snk}(x)$ in (4.14) is evaluated by the formula

$$G_{snk}(x) = -2i(ix\pi)^{1/2} D(z_s) \varphi_{nk}(z_s) \psi_{nk}^{-3/2}(z_s) [(\partial F_1 / \partial z)^{-1} F_2]_{\substack{\lambda=\lambda_{nk} \\ z=z_s}} H(\vartheta_{nk} - \vartheta_s) \tag{4.17}$$

$$\vartheta_s = -h [\operatorname{Re}(z_s)]$$

Here the quantity ϑ_s is determined by using (4.3) from the condition that the saddle point is under the pole with coordinate z_s. The functions $p_0{}^F$, p^F_{snk} in (4.12) are components of the stationary echo, i.e. the ecno signal originating during incidence of the stationary spherical wave

$$p_i = A_0 l^{-1} e^{-ix(\tau - l)}$$

on the shell.

The function $p_0{}^F$ determines the echo reflected from the shell surface by the laws of geometric optics, and the rest p^F_{snk} is the echo radiated by the circumferential and creeping waves being propagated respectively in $(k = 1)$ and opposite to $(k = 2)$ the direction of the polar angle ϑ. To calculate the radiated component of the echo signal, the contributions from the diverse modes (in s) of the circumferential and creeping waves and the contributions from the different rotations (in n) of these waves around the shell are summed in (4.12). The quantities $\psi_0(z_0)$, $\psi_{nk}(z_s)$ in (4.14) determine the time elapsed on the path from the source to the observation point, where $d(z_0)$, $d(z_s)$ determine the times elapsed in propagation from the source to the shell in the fluid and from it to the observation point, and $h(z_s) + \vartheta_{nk}$ on the path in the shell or in the fluid around the shell.

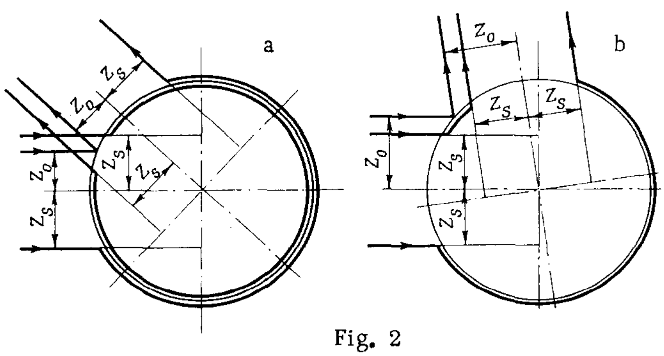

Fig. 2

Figure 2 shows the projections on the $\xi = 0$ plane of paths from the source to the observation point for the reflected wave and one mode of the circumferential waves in ($k = 1$) and opposite to ($k = 2$) the directions of the polar angle ϑ for $r \gg 1, r_0 \gg 1$. Figure 2a shows the paths for $p_0{}^F$, p_{sn1}^F for $n = 1$ and p_{sn2}^F for $n = 0$ in the case $\vartheta < \vartheta_s$, and Fig. 2b shows the paths for $p_0{}^F$ and p_{sn1}, p_{sn2} for $n = 0$ in the case $\vartheta > \vartheta_s$.

5. Inversion of the Fourier time transformation. Let us rewrite the echo signal (4.12) as

$$p_e = B_0 \left[I_0 + \sum_{k=1}^{2} \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} I_{snk} \right] \tag{5.1}$$

$$I_0 = \int_{-\infty}^{\infty} g(x) G_0(x) \exp\{-ix[\tau - \psi_0(z_0)]\} dx \tag{5.2}$$

$$I_{snk} = \int_{-\infty}^{\infty} g(x) G_{snk}(x) \exp\{-ix[\tau - \psi_{nk}(z_s)\} dx \tag{5.3}$$

It is proposed to calculate the integrals (5.2), (5.3) by the same method, hence let us examine the inversion of just the integral (5.2).

By using the convolution theorem the integral (5.2) can be represented as

$$I_0 = \frac{1}{2\pi} \int_0^{\tau_p} f(\zeta) T(\tau - \zeta) d\zeta \tag{5.4}$$

$$T(\tau) = \int_{-\infty}^{\infty} \exp[b(x) - ix\tau] dx, \qquad b(x) = \ln G_0(x) + ix\psi_0(z_0) \tag{5.5}$$

Here $f(\tau)$ is the law of pressure variation in the incident pulse (1.4). Let us examine inversion of the integral (5.5) by using the approximation

$$b(x) \approx b_0 + b_1(x - x_0) - b_2(x - x_0)^2 \tag{5.6}$$

where x_0 is the characteristic frequency of the pulse (1.4) incident on the shell. In the absence of such a characteristic frequency, the frequency corresponding to the maximum value of the function $g(x)$ should be taken as x_0. The coefficients b_0, b_1, b_2 in (5.6) can be defined as the coefficients of the Taylor series expansion of the function $b(x)$ at $x = x_0$ or by some other method of approximating $b(x)$. Substituting (5.6) into (5.5), we obtain the following approximate formula for $T(\tau)$ after calculations:

$$T(\tau) = (\pi / b_2)^{1/2} \exp[b_0 - ix_0\tau - (\tau + ib_1)^2 (4b_2)^{-1}] \tag{5.7}$$

If the pressure in the incident pulse (1.4) varies as

$$f(\tau) = e^{-ix_0\tau} \tag{5.8}$$

then by substituting (5.7), (5.8) into (5.4) we obtain

$$I_0 = {}^1/_2 [\Phi(y_0) - \Phi(y_p)] \exp(b_0 - ix_0\tau) \tag{5.9}$$

$$\Phi(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2} du$$

$$y_0 = (\tau + ib_1) 2^{-1} b_2^{-1/2}, \qquad y_p = (\tau + ib_1 - \tau_p) 2^{-1} b_2^{-1/2}$$

In the limit case $b_2 = 0$, $\operatorname{Re} b_1 = 0$, formula (5.9) becomes

$$I_0 = G_0(x_0)\, e^{-i\lambda_0 [\tau - \psi_0(z_0)]} [H(\tau - \operatorname{Im} b_1) - H(\tau - \operatorname{Im} b_1 - \tau_p)]$$

The echo signal (5.1) consists of one reflected echo pulse I_0 and of two ($k = 1, 2$) double ($s = 1, 2, \ldots, n = 0, 1, \ldots$) series of radiated echo pulses I_{snk}. Because of damping the number of radiated pulses whose amplitudes differ from zero in practice turns out not to be larger, and the series in (5.1) converge rapidly.

On the basis of the approximations used and the approximate methods of evaluating the integrals, the use of the proposed method of evaluating the echo signal can be justified only for incident pulses of oscillatory nature, and whose characteristic frequency x_0 is large. On the other hand, on the basis of the hypotheses of Timoshenko-type shell theory, the characteristic frequency x_0 should not be so high that the wavelength of the incident pulse would be commensurate with the shell thickness. Admissible values for the characteristic frequency x_0 can be characterized by the inequalities

$$R_k h^{-1} > x_0 \pi^{-1} > 1$$

where the signs of the inequalities should here be understood in the sufficiently strong sense.

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